A SPECIAL QUADRI-QUADRIC TRANSFORMATION OF REAL POINTS IN A PLANE

By Carl C. Engberg

The transformation here studied is strictly speaking a (2, 2) transformation, for the equations of the direct transformation and of its inverse contain a double valued function, so that every point (x, y) goes over into two points (x', y'), and conversely.

By confining ourselves to real points of the plane, however, and making a suitable convention in regard to the sign of the radical, we reduce these double valued functions to single valued functions, and thus obtain a (1, 1) correspondence between the points of half the plane (viz., the quadrants above and below the lines $y = \pm x$) and the points of the whole plane.

For a general discussion of (2, 2) transformations the reader should consult articles by P. Visalli* and Burali-Forti.† The special transformation here studied has been employed by the writer in a paper on the Cartesian Oval,‡ where many properties of these ovals are obtained by applying the transformation to a parabola.

1. *Definition of the Transformation.* The transformation here studied is defined by the equations

$$\left.\begin{aligned} x &= x' \\ y &= \pm \sqrt{x'^2 + y'^2} \end{aligned}\right\} \text{whence} : \left\{\begin{aligned} x' &= x \\ y' &= \pm \sqrt{y^2 - x^2}, \end{aligned}\right.$$

whereby we agree that the sign of y' shall be the same as the sign of y.

If the given point (x, y) is real, the transformed point (x', y') will be real or imaginary according as $x < y$ or $x > y$.

Geometrically speaking, the transformed point P is that point whose abscissa is the abscissa of the given point P, and whose radius vector is the ordinate of P. If the abscissa of the given point is less than its ordinate, we can construct two such points P' but we agree to take that one which lies on the

* *Rendiconti del Circolo Matematico di Palermo*, vol. 3 (1889), pp. 165-170.

† *Ibid.*, vol. 5 (1891), pp. 91–99.

‡ *Graduate Bulletin of the University of Nebraska*, vol. 1 (1900), pp. 23–40.

same side of the axis of x as P does. If the abscissa of the given point is greater than its ordinate, the transformed point does not exist.

2. *Resulting Deformation of the Plane.* By this transformation a straight line through the origin, say $y = mx$, is carried over into another straight line through the origin, viz:

$$y' = \pm \sqrt{m^2 - 1}\, x',$$

where the sign of the radical is to be taken the same as the sign of m. When the inclination of the given line varies from 90° to 45° ($\infty > m > 1$), the inclination of the transformed line varies from 90° to 0°; when it becomes less than 45°($1 > m > 0$), the transformed line becomes imaginary.

The effect of the transformation may then be described as follows (confining ourselves to the upper half of the plane): the quadrantal region* above the lines $y = \pm x$ is expanded like a fan until its bounding lines coincide with the axis of x; the points of the axis of y remain fixed, while all other points of the region move towards the x-axis along lines perpendicular to that axis. The portion of the plane between the lines $y = \pm x$ and the x-axis becomes imaginary.

3. *Properties of the Transformation* (A).

If a curve is symmetrical with respect to the x-axis, its transformed curve will also be symmetrical with respect to that axis.

If two curves are tangent to each other at a given point, the corresponding curves will also be tangent to each other at the corresponding point.

The principal portion of a straight line $x = a$ perpendicular to the x-axis is transformed into the whole line $x = a$, the two points $(a, \pm a)$ uniting in the single point $(a, 0)$.

The principal portion of a straight line $y = b$ parallel to the x-axis goes over into the semicircle $x^2 + y^2 = b^2$; the pair of lines $y = \pm b$ gives the whole circle.

The principal portion of any straight line $y = mx + b$ is transformed into two quarters of an x-symmetric conic

$$y^2 = (m^2 - 1)x^2 + 2bmx + b^2,$$

* We shall speak of this region, together with the corresponding region in the lower half of the plane, as forming the "principal portion" of the plane; the "principal portion" of any curve shall then mean that portion of the curve which lies in the principal portion of the plane.

while the pair of lines $y = \pm (mx + b)$ gives the whole conic. The conic is tangent to the given lines where they cross the y-axis; one of its foci is at the origin; the corresponding directrix meets the given line on the x-axis; and the eccentricity is m. The conic will be an ellipse, parabola, or hyperbola according as the inclination of the given line is $< 45°$, $= 45°$, or $> 45°$.

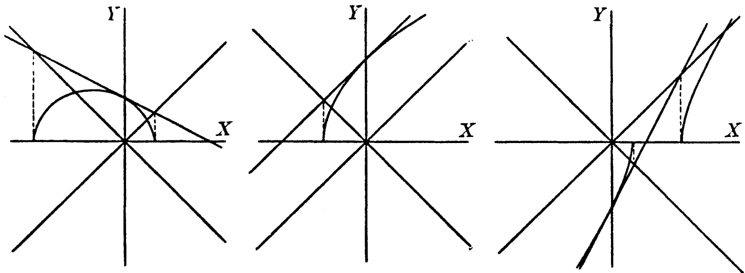

FIG. 1.

A set of parallel lines goes over into a set of x-symmetric conics having a constant eccentricity and a common focus at the origin.

The parabola $y^2 = 2ax$ goes over into the circle $x^2 + y^2 = 2ax$.

The parabola $y^2 - 2Ay - 2Bx + C^2 = 0$ goes over into the Cartesian Oval $\rho^2 - 2A\rho - 2Bx + C^2 = 0$.

An x-symmetric conic goes over into an x-symmetric conic, and if the centre is at the origin it remains there.

A quartic symmetric with respect to the x-axis goes over into another quartic having the same property.

The equilateral hyperbola $y^2 - x^2 + 2gx + 2fy + c = 0$ goes over into the quartic $y^2 + 2gx + 2f\rho + c = 0$, which in its turn goes over into a bicircular quartic.

4. *The Inverse Transformation* (B). The inverse transformation will clearly transform the whole plane into half the plane—viz. the half which lies above and below the lines $y = \pm x$. The following properties of the inverse transformation, which we shall designate as transformation B, will aid in giving a conception of it:

A straight line $y = mx + b$ is transformed by B into parts of an x-symmetric hyperbola

$$y^2 = (m^2 + 1)x^2 + 2bmx + b^2,$$

while the pair of lines $y = \pm (mx + b)$ gives the whole hyperbola. The hy-

perbola will lie wholly in the principal portion of the plane; it will be tangent to the lines $y = \pm x$, and also to the given lines. The slopes of its asymptotes will be $\pm \sqrt{m^2 + 1}$.

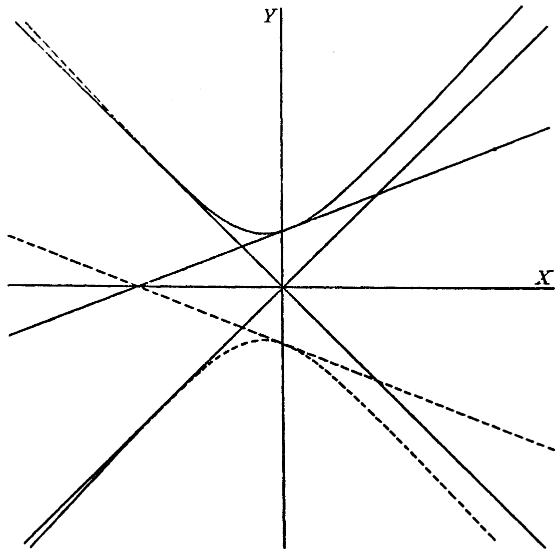

FIG. 2.

The circle $x^2 + y^2 = 2ax$ goes over into the parabola $y^2 = 2ax$.
The circle $(x - a)^2 + y^2 = r^2$ is transformed into the parabola

$$y^2 = 2ax + r^2 - a^2.$$

In general, B transforms an x-symmetric conic into another x-symmetric conic, and if the centre is at O it will remain there. If the focus of the conic is at O, the conic goes over into a pair of straight lines.

The limaçon $\rho = 2a \cos\theta \pm c$ is transformed into two equal parabolas

$$(y \pm c/2)^2 = 2ax + c^2/4.$$

The conchoid $a = (\rho \pm c) \cos \theta$ is transformed into two equal equilateral hyperbolas $xy = ay \pm cx$.

The cissoid $\rho = 2a(\sec \theta - \cos \theta)$ goes over into the cubic

$$xy^2 + 2a(x^2 - y^2) = 0.$$

5. *Applications.* We now proceed to derive a few theorems from

known theorems on conics and straight lines by means of the direct transformation (A) or its inverse (B).

1. "A variable line through O cuts the line $x = a$ in P, and Q is taken on this variable line so that $OP \times OQ$ is constant; the locus of Q is a circle through O."

Transforming by (B) we have:

A variable line through O cuts the line $x = a$ in P, and Q is taken on this variable line so that the product of the ordinates of P and Q is constant; the locus of Q is a parabola through O.

2. "A variable line through O cuts the circle $(x - a)^2 + y^2 = p^2$ in P, and Q is taken on this line so that $OP \times OQ$ (or $OP \div OQ$) is constant; the locus of Q is an x-symmetric circle."

Transforming by (B), we have:

A variable line through O cuts an x-symmetric parabola in P, and Q is taken on this line so that the product (or the quotient) of the ordinates of P and Q is constant; the locus of Q is another x-symmetric parabola.

3. "A variable line which moves always parallel to itself cuts two fixed lines in P and Q; the locus of the middle point of PQ is a straight line through the point of intersection of the two fixed lines."

Transforming by (A) we have:

A variable x-symmetric conic, having a focus at O and a constant eccentricity, meets two fixed x-symmetric conics having a common focus at O in the points P and Q on the same side of the axis. On the variable conic a point is taken whose abscissa is the arithmetic mean of the abscissas of P and Q; then the locus of this point is another x-symmetric conic having a focus at O and passing through the points of intersection of the two fixed conics.

Transforming by (B) we have another theorem obtained from this by changing the word "conic" to "hyperbola," and the words "having a focus at O" to the words "tangent to the lines $y = \pm x$."

4. "A variable line through O meets the $\left\{\begin{array}{l}\text{circle } x^2 + y^2 = 2ax \\ \text{line } x = a\end{array}\right\}$ in P, and Q is taken on this variable line at a constant distance $\pm c$ from P. Then the locus of Q is the $\left\{\begin{array}{ll}\text{limaçon} & \rho = 2a \cos \theta \pm c \\ \text{conchoid} & a = (\rho + c) \cos \theta\end{array}\right\}$."

Transforming by (B) we have:

A variable line through O meets the $\left\{\begin{array}{l}\text{parabola } y^2 = 2ax \\ \text{line } x = a\end{array}\right\}$ in P, and

Q is taken on this variable line so that the difference between the ordinates of P and Q is a constant, $\pm c$. Then the locus of Q is two equal $\left\{\begin{matrix}\text{parabolas} \\ \text{equilateral hyperbolas}\end{matrix}\right\}$ through O.

5. "In the parabola $y^2 = 4ax$ let P, Q, R be points whose ordinates are in geometric progression; then the tangents at P and R meet on the ordinate of Q."

Transforming by (A) we have:

In the circle $x^2 + y^2 = 4ax$ let P, Q, R be three points whose radii vectores are in geometric progression; then the two x-symmetric conics having a common focus at O and touching the circle, the one at P and the other at R, will meet on the ordinate of Q.

6. "A variable line through O meets the fixed circle $x^2 + y^2 = 2ax$ in P and the fixed tangent $x = 2a$ in Q. On this variable line take $OR = PQ$; then the locus of R is the cissoid $\rho = 2a(\sec\theta - \cos\theta)$."

Transforming by (A) we have:

A variable line through O meets the fixed parabola $y^2 = 2ax$ in P and the fixed line $x = 2a$ in Q. On this variable line the point R is taken whose ordinate is the difference between the ordinates of P and Q; then the locus of R is the cubic $xy^2 + 2a(x^2 - y^2) = 0$.

In conclusion we may note that the transformation may be easily extended to three dimensions, the equations of the transformation being

$$x = x', \quad y = y', \quad z = \pm\sqrt{x'^2 + y'^2 + z'^2}.$$

Here the sign of the radical is to be taken the same as the sign of z; the "principal portion" of space will then be the region above the four planes $z = \pm x$, $z = \pm y$, together with the corresponding region below the xy-plane.

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